

# The Hydrogen Spectrum in Non-Commutative Space-Time: Application to the Lyman- $\alpha$ Line and the $2S - 1S$ Transition.

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## Abstract

Recently, there has been a certain amount of activity around the theme of cosmological and astrophysical applications of noncommutative geometry models of particle physics. We study space-time noncommutativity applied to the hydrogen atom and the corrections induced to transitions frequencies. By solving the Dirac equation for noncommutative Coulomb potential, we compute noncommutative corrections of the energy levels and by comparing to the Lamb shift accuracy we get a bound on the parameter of noncommutativity. We use this bound to study the effects on the the Lyman- $\alpha$  ray and the  $2S - 1S$  transition and by induction the possible influence of noncommutativity on some astrophysical and cosmological phenomena.

## 1 Introduction:

Recently, there has been a certain amount of activity around the theme of cosmological and astrophysical applications of noncommutative geometry models of particle physics [1-7]. This work is in the same context and we are interested in the effects of space-time noncommutativity on the hydrogen spectroscopy and especially on the Lamb shift, the Lyman- $\alpha$  line and the  $2S - 1S$  transition.

The Lyman- $\alpha$  emission line is produced by recombination of electrons with ionized hydrogen atoms. The Lyman- $\alpha$  line and related diagnostics are now

routinely used in numerous astrophysical studies for example to probe massive star formation across the Universe, to study properties of the interstellar and intergalactic medium across cosmic times, and to search for the most distant galaxies in the Universe [8]. Both observations and simulations have made enormous progress over the last decade. For example, surveys targeting Lyman- $\alpha$  have discovered new populations of galaxies out to the highest red shift, and sophisticated hydrodynamic and radiation transfer simulations have been developed.

In astronomy, a Lyman-Alpha Blob (LAB) is a huge concentration of a gas emitting the Lyman- $\alpha$  emission line [8]. LABs are some of the largest known individual objects in the Universe. Some of these gaseous structures are more than 400,000 light years across. So far they have only been found in the high-redshift universe because of the ultraviolet nature of the Lyman- $\alpha$  emission line. Since the Earth's atmosphere is very effective at filtering out UV photons, the Lyman- $\alpha$  photons must be redshifted in order to be transmitted through the atmosphere.

The most famous LABs were discovered in 2000 by Steidel et al.[9]. It is currently unknown whether LABs trace over densities of galaxies in the high-redshift universe (as high redshift radio galaxies, which also have extended Lyman- $\alpha$  halos, do, for example), nor which mechanism produces the Lyman- $\alpha$  emission line, or how the LABs are connected to the surrounding galaxies. Lyman- $\alpha$  Blobs may hold valuable clues for scientists to determine how galaxies are formed.

In astronomical spectroscopy, the Lyman- $\alpha$  forest is the sum of absorption lines arising from the Lyman- $\alpha$  transition of the neutral hydrogen in the spectra of distant galaxies and quasars [10]. These absorption lines result from intergalactic gas through which the galaxy or quasar's light has travelled. The forest is created by the fact that photons that come to us from distant light sources show Hubble redshift that depends on the distance between us and the source of light. Since neutral hydrogen clouds at different positions between Earth and the distant light source see the photons at different wavelengths (due to the redshift), each individual cloud leaves its fingerprint as an absorption line at a different position in the spectrum as observed on Earth.

The Lyman- $\alpha$  forest is an important probe of the intergalactic medium and can be used to determine the frequency and density of clouds containing neutral hydrogen, as well as their temperature. Searching for lines from other elements like helium, carbon and silicon (matching in redshift), the abundance of heavier elements in the clouds can also be studied.

The  $2S - 1S$  transition is used in high precision spectroscopy because of the implication of these measurements on the values of fundamental physical constants like the fine structure constant  $\alpha$  and the Rydberg constant  $R$  [11]. The possible variation of the fine structure constant has relation with primordial light nuclei abundances in the early universe [12], with  $f(R)$  theories in Einstein frame and quintessence models [13] or with the inhomogeneity of the mass distribution in the early universe and the cosmological constant [14]. One can find a good review in [14] or [15].

The idea of taking space-time coordinates to be noncommutative goes back to the thirties of the last century. The goal was that the introduction of a non-commutative structure to space-time at small length scales could introduce an effective cut off which regularize divergences in quantum field theory. However this theory was plagued with several problems such as the violation of unitarity and causality, which make people abandon it. However noncommutative geometry was pursued on the mathematical side and especially with the work of Connes in the eighties of the last century [16].

In 1999, the interest for noncommutative geometry is renewed by the work of Seiberg and Witten on string theory [17]. They showed that the dynamics of the endpoints of an open string on a D-brane in the presence of a magnetic background field can be described by a Yang-Mills theory on a noncommutative space-time.

Noncommutative space-time is a deformation of the ordinary one in which the coordinates are promoted to Hermitian operators which do not commute:

$$[x_{nc}^\mu, x_{nc}^\nu] = i\theta^{\mu\nu} = iC^{\mu\nu} / (\Lambda_{nc})^2; \mu, \nu = 0, 1, 2, 3 \quad (1)$$

where  $\theta^{\mu\nu}$  is a deformation parameter and  $nc$  indices denote noncommutative coordinates. Ordinary space-time is obtained by making the limit  $\theta^{\mu\nu} \rightarrow 0$ . The noncommutative parameter is an anti-symmetric real matrix.  $\Lambda_{nc}$  is the energy scale where the noncommutative effects of the space-time will be relevant and  $C^{\mu\nu}$  are dimensionless parameters. A well documented review can be found in [18-19].

We are interested in the noncommutative effects on the hydrogen atom spectroscopy. We start by writing the Dirac equation for the H-atom in framework of noncommutative space-time. Then, we compute the corrections of the energy levels induced by noncommutativity and get a limit on the noncommutative parameter from the  $2P - 2S$  Lamb shift (or the  $28cm$  line) theoretical accuracy. We use this bound to estimate the contribution to the Lyman- $\alpha$  line. We also study the effects on high precision spectroscopy of hydrogen via the  $2S - 1S$  transition frequency measurement.

## 2 Noncommutative Hydrogen Atom Spectrum:

We work here on the space-time version of the noncommutativity; thus instead of (1), we use:

$$[x_{st}^j, x_{st}^0] = i\theta^{j0} \quad (2)$$

the  $st$  subscripts are for noncommutative space-time coordinates. The 0 denotes time and  $j$  is used for space coordinates. As a solution to these relations, we choose the transformations:

$$x_{st}^j = x^j + i\theta^{j0}\partial_0 \quad (3)$$

The usual coordinates of space  $x^j$  satisfy the usual canonical permutation relations. For convenience we use the vectorial notation:

$$\vec{r}_{st} = \vec{r} + i \vec{\theta} \partial_0 ; \quad \vec{\theta} \equiv (\theta^{10}, \theta^{20}, \theta^{30}) \quad (4)$$

The relations (3) and (4) can be seen as a Bopp's shift.

We write the Dirac equation:

$$i\hbar\partial_0 = H\psi = (\vec{\alpha} \cdot \vec{p}) + m\gamma^0 - eA_0 \quad (5)$$

where  $\alpha_i = \gamma_0\gamma_i$  and  $\gamma_\mu$  are the Dirac matrices.

As the kinetic energy depends on the momentum  $\vec{p}$  which remains unchanged, it does not change; that is why we consider only the potential energy by taking the Coulomb potential and constructing its noncommutative image. To achieve this, we write it as the usual one but with the new coordinates:

$$A_0^{(nc)} = \frac{e}{r_{st}} = e \left[ \left( \vec{r} + i \vec{\theta} \partial_0 \right)^2 \right]^{-1/2} \quad (6)$$

Because of the smallness of the noncommutative parameter [18], we restrict ourselves to the 1st order in  $\theta$  and neglect the higher order terms in the development in series of the expression and write:

$$A_0^{(nc)} = \frac{e}{r} \left( 1 - \frac{i\partial_0 \vec{r} \cdot \vec{\theta}}{r^2} + O(\theta^2) \right) \quad (7)$$

An adequate choice of the parameter is  $\vec{\theta} = \theta^{r0} \vec{r}/r$ ; It is equivalent to that written in [6] in the case of noncommutative space-space and in [7] for the space-time case. This choice allows us to write the noncommutative Coulomb potential as (we note  $\theta^{r0} = \theta_{st}$ ):

$$A_0^{(nc)} = \frac{e}{r} \left( 1 - i\partial_0 \frac{\theta_{st}}{r} \right) + O(\theta^2) = \frac{e}{r} - \frac{eE\theta_{st}}{\hbar} \frac{1}{r^2} + O(\theta^2) \quad (8)$$

where we have used the fact that  $i\partial_0\psi = (E/\hbar)\psi$ . The Hamiltonian can now be expressed as:

$$H = (\vec{\alpha} \cdot \vec{p}) + m\gamma^0 - e \left( e/r - e(E/\hbar)\theta_{st}/r^2 \right) = H^{(0)} + H^{(nc)} \quad (9)$$

$H^{(0)}$  is the Dirac Hamiltonian in the usual relativistic theory and  $H^{(nc)}$  is the noncommutative correction to this Hamiltonian:

$$H^{(nc)} = e^2 (E/\hbar) \theta_{st} r^{-2} \quad (10)$$

The smallness of the parameter  $\theta$  allows us to consider noncommutative corrections with perturbation theory; to the 1st order in  $\theta$ , the corrections of the eigenvalues are:

$$E^{(nc)} = \langle H^{(nc)} \rangle = Ee^2/\hbar \langle r^{-2} \rangle \theta_{st} \quad (11)$$

From [20], one has:

$$\left\langle \frac{1}{r^2} \right\rangle = \frac{2\kappa(2\kappa\varepsilon - 1)(1 - \varepsilon^2)^{3/2}}{\alpha\sqrt{\kappa^2 - \alpha^2}[4(\kappa^2 - \alpha^2) - 1]} \left(\frac{mc}{\hbar}\right)^2 \quad (12)$$

where  $a_0 = \hbar^2/mc^2$  is the 1st Bohr radius and  $\varepsilon = E/mc^2$ .  $E$  is the Dirac energy:

$$E = mc^2 \left\{ 1 + \alpha^2 \left[ (n - j - 1/2) + \sqrt{(j + 1/2)^2 - \alpha^2} \right]^{-2} \right\}^{-1/2} \quad (13)$$

$\alpha = e^2/\hbar c$  is the fine structure constant and  $j = l \pm 1/2$  is the quantum number associated to the total angular momentum  $\vec{j} = \vec{l} + \vec{s}$ . The number  $\kappa$  is giving by the two relations:

$$j = (l + 1/2) \Rightarrow \kappa = -(j + 1/2) \quad (14a)$$

$$j = (l - 1/2) \Rightarrow \kappa = (j + 1/2) \quad (14b)$$

We see from (11) and (12) that through  $\kappa$ , the energy depends not only on the value of  $j$  but also on the manner to get this value (or on  $l$ ), unlike the usual Dirac energies in (13) which is the same for all the possible ways to obtain  $j$ ; It implies that the noncommutativity remove the degeneracy  $j = l + 1/2 = (l + 1) - 1/2$  in hydrogen atom (in states like  $nP_{3/2}$  and  $nD_{3/2}$ ).

We recall that the energy level without considering the rest mass energy ( $mc^2$ ) is written as a function of the total energy by the relation  $E_{n,j} = E - mc^2$  and so the corrections to these energies are:  $\Delta E_{n,j}^{(nc)} = \Delta E^{(nc)}$ . From now on, we note these corrections  $E_{n,j}^{(nc)}$  or  $E^{(nc)}(nL_j)$  where  $L$  is the spectroscopic letter corresponding to a specific value of the angular quantum number  $l$ . As an example, we compute the corrections to the levels  $n = 1, 2$ :

$$E^{(nc)}(1S_{1/2}) = 1.065084 \cdot 10^{-4} (m^3 e^2 c^4 / \hbar^3) \theta_{st} \quad (15a)$$

$$E^{(nc)}(2S_{1/2}) = 1.331426 \cdot 10^{-5} (m^3 e^2 c^4 / \hbar^3) \theta_{st} \quad (15b)$$

$$E^{(nc)}(2P_{1/2}) = 0.443805 \cdot 10^{-5} (m^3 e^2 c^4 / \hbar^3) \theta_{st} \quad (15c)$$

$$E^{(nc)}(2P_{3/2}) = 0.443765 \cdot 10^{-5} (m^3 e^2 c^4 / \hbar^3) \theta_{st} \quad (15d)$$

To compare with the usual Dirac energies, we make the development of the corrections to the 2nd order of  $\alpha$ . For the Dirac energies, we have [21]:

$$E = mc^2 \left\{ 1 - \frac{\alpha^2}{2n^2} \left[ 1 + \left( \frac{2}{(2j+1)n} - \frac{3}{4n^2} \right) \alpha^2 \right] + O(\alpha^6) \right\} \quad (16)$$

For the noncommutative correction to this expression, we find two expressions depending on the value of  $\kappa$ :

$$E_{n,j=l+\frac{1}{2}}^{(nc)} = \frac{m^3 e^2 c^4 \alpha^2}{jn^3 \hbar^3} \left[ 1 + \left( \frac{6j^2+6j+1}{j(j+1)(2j+1)^2} + \frac{3}{(2j+1)n} - \frac{10j+9}{4(j+1)n^2} \right) \alpha^2 \right] \theta_{st} \quad (17a)$$

$$E_{n,j=l-\frac{1}{2}}^{(nc)} = \frac{m^3 e^2 c^4 \alpha^2}{(j+1)n^3 \hbar^3} \left[ 1 + \left( \frac{6j^2+6j+1}{j(j+1)(2j+1)^2} + \frac{3}{(2j+1)n} - \frac{10j+1}{4jn^2} \right) \alpha^2 \right] \theta_{st} \quad (17b)$$

We use the general expressions from (11), (12) and (13) to compute the transition energy and make the development in series with respect to  $\alpha$  to get the same result. We see that the expressions depend on way to obtain the value of  $j$  unlike the usual Dirac energies in (13) or (16) which is the same for all the possible values of  $j$  as we have mentioned before. It implies that the noncommutativity acts like a Lamb shift and remove the degeneracy  $j = l + 1/2 = (l + 1) - 1/2$  in the hydrogen. From the precedent expressions, we write the noncommutative correction to the Lamb Shift:

$$\Delta E_{n,j}^{(nc)}(Lamb\ shift) = E_{n,j=l+1/2}^{(nc)} - E_{n,j=(l+1)-1/2}^{(nc)} \quad (18a)$$

$$= \frac{m^3 e^2 c^4 \alpha^2}{j(j+1)n^3 \hbar^3} \left[ 1 + \left( \frac{6j^2 + 6j + 1}{j(j+1)(2j+1)^2} + \frac{3}{(2j+1)n} - \frac{2}{n^2} \right) \alpha^2 \right] \theta_{st} \quad (18b)$$

We compute now the correction to the  $n = 2$  and  $j = 1/2$  case or the  $2P_{1/2} \rightarrow 2S_{1/2}$  Lamb shift. From (18) (or from the general relation which can be easily be written (11) and (12)), we have:

$$\begin{aligned} \Delta E_{2,1/2}^{(nc)}(Lamb\ shift) &= E^{(nc)}(2P_{1/2}) - E^{(nc)}(2S_{1/2}) \\ &= 0.887621 \cdot 10^{-5} (m^3 e^2 c^4 / \hbar^3) \theta_{st} \end{aligned} \quad (19)$$

We compare this result to the current theoretical accuracy for the Lamb shift  $0.08\text{ kHz}$  from [22] and find the bound:

$$\theta_{st} \lesssim 3.254 \cdot 10^{-23} \text{ eV}^{-2} \approx (0.18 \text{ TeV})^{-2} \quad (20)$$

It is better than those obtained in [19] and [23] (It is different from the correction in the space-space case of noncommutativity in [24] where the correction depends on the quantum number  $m$  while it is independent of the value of  $m$  in our case, because our choice for the parameter  $\theta$  preserves the spherical symmetry of the potential).

We use this limit to see the effects on the Lyman- $\alpha$  line. The noncommutative corrections to this ray are:

$$\begin{aligned} \Delta E^{(nc)}(Ly\alpha_1) &= E^{(nc)}(2P_{1/2}) - E^{(nc)}(1S_{1/2}) \\ &= 1.020703 \cdot 10^{-4} (m^3 e^2 c^4 / \hbar^3) \theta_{st} \end{aligned} \quad (21a)$$

$$\begin{aligned} \Delta E^{(nc)}(Ly\alpha_2) &= E^{(nc)}(2P_{3/2}) - E^{(nc)}(1S_{1/2}) \\ &= 1.020707 \cdot 10^{-4} (m^3 e^2 c^4 / \hbar^3) \theta_{st} \end{aligned} \quad (21b)$$

Using the value of  $\theta$  found in (20), we have:

$$\begin{aligned} \Delta E^{(nc)}(Ly\alpha_1) &= 3.8045851 \times 10^{-12} \text{ eV} \\ \Rightarrow \Delta \lambda^{(nc)}(Ly\alpha_1) &= 4.530013 \times 10^{-10} \text{ \AA} \end{aligned} \quad (22a)$$

$$\begin{aligned} \Delta E^{(nc)}(Ly\alpha_2) &= 3.804600 \times 10^{-12} \text{ eV} \\ \Rightarrow \Delta \lambda^{(nc)}(Ly\alpha_2) &= 4.529990 \times 10^{-10} \text{ \AA} \end{aligned} \quad (22b)$$

These corrections are somewhat  $10^{-13}$  smaller than the value of the Lyman- $\alpha$  wavelength ( $1215.67 \text{ \AA}$ ), but the ratio is just about 1/3 comparing to the accuracy of the line which is  $1.18 \times 10^{-9} \text{ \AA}$  from the physical data of the National Institute of Standards and Technology [25]. We conclude that the effect of noncommutative space-time on the Lyman- $\alpha$  line is negligible up to now.

We study now the effects on high precision spectroscopy of the H-atom. We take as test levels,  $1S$  and  $2S$  because we have the best experimental precision for the transition between them [11]:

$$\nu_{1S-2S} = (2446061102474851 \pm 34) \text{ Hz} \quad (23)$$

From (15), the noncommutative correction for this transition is:

$$\Delta E^{(nc)}(1S - 2S) = 0.931941 \cdot 10^{-4} (m^3 e^2 c^4 / \hbar^3) \theta_{st} \quad (24)$$

Using (20), we get:

$$\Delta E^{(nc)}(1S - 2S) = 3.473732 \cdot 10^{-12} \text{ eV}^{-2} \quad (25a)$$

$$\Rightarrow \Delta \nu^{(nc)}(1S - 2S) = 0.840 \text{ kHz} \quad (25b)$$

We see that this correction is greater than the precision of the experimental value in (23) and thus noncommutativity can be considered in high precision spectroscopy and optical frequency metrology; it can affect the accurate determination of fundamental constants such as the fine structure constant  $\alpha$  and the Rydberg constant  $R$  (if we take the bound of the noncommutative parameter from the  $28cm$  line).

### 3 Conclusion:

In this work, we look for space-time noncommutative hydrogen atom and induced phenomenological effects; for this we use the Bopp's shift formulation. We found that applying space-time noncommutativity to the electron in the H-atom modifies the Coulomb potential to give us the potential of Kratzer.

By solving the Dirac equation, we have calculated the corrections induced to energy levels. The space-time noncommutative corrections to the Dirac theory of hydrogen atom remove the degeneracy of the Dirac energies with respect to the total angular momentum quantum number  $j = l + 1/2 = (l + 1) - 1/2$  in addition to the degeneracy of the Bohr energies with respect to the orbital quantum number  $l$ , and the energies are labelled  $E_{n,j,l}$ . The noncommutativity acts like a Lamb shift here. This is explained by the fact that Lamb correction can be interpreted as a shift of  $r$  in the Coulomb potential due to interactions of the bound electron with the fluctuating vacuum electric field [21], and noncommutativity is also a shift of  $r$  as we can see from the Bopp's shift.

By comparing to theoretical limit of the Lamb shift, we get a bound on the parameter of noncommutativity. We use this limit to see the noncommutative effects on the Lyman- $\alpha$  line and found that they are about 34% smaller than

the actual experimental accuracy; the noncommutative corrections in this case are negligible up to now.

In another application of the bound from the Lamb shift, we compute the effects on the  $2S - 1S$  transition. In this case, the corrections are found to be greater than the experimental accuracy from high precision spectroscopy. Because of the relation of this transition measurement with the values of fundamental physical constants, the noncommutativity can have appreciable effects on the possible variation of the fine structure constant and by induction on the cosmological constant, on theories of gravitation and on some characteristics of the early universe like primordial light nuclei abundances or mass distribution.

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